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A short review on entanglement in quantum spin systems

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Abstract

We review some of the recent progress on the study of entropy of entanglement in many-body quantum systems. Emphasis is placed on the scaling properties of entropy for one-dimensional multi-partite models at quantum phase transitions and, more generally, on the concept of area law. We also briefly describe the relation between entanglement and the presence of impurities, the idea of particle entanglement, the evolution of entanglement along renormalization group trajectories, the dynamical evolution of entanglement and the fate of entanglement along a quantum computation.

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1. Introduction

Quantum systems are ultimately characterized by the observable correlations they exhibit. For instance, an observable such as the correlation function between two spins in a typical spin chain may decay exponentially as a function of the distance separating them or, in this case the system undergoes a phase transition, algebraically. The correct assessment of these quantum correlations is tantamount to understanding how entanglement is distributed in the state of the system. This is easily understood as follows. Let us consider a connected correlation

$$\langle \Psi | O_i O_j | \Psi \rangle_c \equiv \langle \Psi | O_i O_j | \Psi \rangle - \langle \Psi | O_i | \Psi \rangle \langle \Psi | O_j | \Psi \rangle, \quad (1)$$

where O_i and O_j are operators at sites i and j , respectively. This connected correlator would vanish identically for any product state $|\Psi\rangle = \otimes_i |\psi_i\rangle$. That is, $O_i \otimes O_j$ is a product operator and, consequently, its correlations can only come from the amount of entanglement in the state $|\Psi\rangle$. It follows that the ground state of any interesting system will be highly correlated and, as a particular case, even the vacuum displays a non-trivial entanglement structure in quantum field theories.

Note that, at this point, our emphasis has moved from Hamiltonians to states. It is perfectly sensible to analyse the entanglement properties of specific states per se, which may be artificially created using a post-selection mechanism or may effectively be obtained in different

ways using various interactions. We are, thus, concerned with the entanglement properties that characterize a quantum state. Yet, we shall focus on states that are physically relevant. In particular, we shall study the entanglement properties of ground states of Hamiltonians that describe the interaction present in spin chains.

It is clear that the property of entanglement can be made apparent by studying correlation functions on a given state. We could consider two-, three- or n -point connected correlation functions. Any of them would manifest how the original interactions in the Hamiltonian have operated in the system to achieve the observed degree of entanglement. For instance, free particles (Gaussian Hamiltonians) produce n -point correlators that reduce to products of two-point correlators via Wick's theorem. Nonetheless, the study of specific correlation functions is model dependent. How can we compare the correlations of a Heisenberg model with those of quantum chromodynamics? Each theory brings its own set of local and non-local operators that close an operator product expansion. Different theories will carry different sets of operators, so that a naive comparison is hopeless. A wonderful possibility of quantifying degrees of entanglement for unrelated theories emerges from the use of renormalization group ideas and the study of universal properties. For instance, a system may display exponential decays in its correlation functions which is globally controlled by a common correlation length. A model with a larger correlation length is expected to present stronger long-distance quantum correlations.

We will also try to find a universal unique figure of merit that would allow for a fair comparison of the entanglement present in e.g. the ground state of all possible theories. Such a figure of merit cannot be attached to the correlation properties of model-dependent operators since it would not allow for comparison among different theories. The way to overcome this problem is to look for an operator which is defined in every theory. It turns out that there is only one such operator: the stress tensor. To be more precise, we can use the language of conformal field theory which establishes that there is always a highest weight operator that we call the identity. The identity will bring a tower of descendants, the stress tensor being its first representative. Indeed, the stress tensor is always defined in any theory since it corresponds to the operator that measures the energy content of the system and it is the operator that couples the system to gravity. Correlators of stress tensor operators are naturally related to entanglement. In particular, the coefficient of the two-point stress tensor correlator in a conformal field theory in two dimensions corresponds to the central charge of the theory.

There is a second option to measure entanglement in a given state with a single measure of entanglement which is closer in spirit to the ideas of quantum information. The basic idea consists of using the von Neumann (entanglement) entropy of the reduced density matrix of a sub-part of the system which is analysed. Indeed, the entanglement entropy quantifies the amount of surprise that a sub-part of a system finds when discovering that it is correlated with the rest of the system. Therefore, entanglement entropy is a *bona fide* measure of the correlations in the system. The advantage of the von Neumann entropy of entanglement is that it can be defined for any system. We expect these general properties, due to the way it scales with the size of the sub-part of the system we are considering, should be characterized by the quantum state in a quite refined way.

It is tantalizing to exhaustively explore the behaviour of the entropy of entanglement in relevant physical systems. For instance, will the entropy of entanglement scale differently at a critical point as compared to a non-critical one? Will scaling properties depend on the dimensionality of the system. Is disorder relevant for long-distance correlations? Are there non-local systems where entropy obeys some singular behaviour? How does entanglement renormalize? How does entanglement evolve dynamically? We can even go further away

from standard dynamical models and question whether entanglement is somehow related to computational complexity problems, both NP-complete and QMA-complete. We shall now briefly review some of these questions.

2. An explicit computation of entanglement entropy

Let us start our discussion with the study of the behaviour of entanglement at different regimes (critical and non-critical) of the XX model. As we shall see, entanglement entropy will be a suitable tool to describe the properties of the quantum phase transition which characterize this model [1, 2].

In order to do this, first we need to introduce the von Neumann entropy as a measure of the bipartite entanglement in pure states. Then we will study the scaling of the entanglement entropy for the simple XX model. We will then proceed to compute the ground state $|GS\rangle$ of the system, from which we can obtain the spectra of the reduced density matrix ρ_L for the block of L contiguous spins. The knowledge of the eigenvalues of ρ_L will let us determine its entanglement entropy S_L . Finally, we are going to analyse how the entanglement behaves depending on the critical properties of the model.

2.1. Entanglement entropy

The problem of measuring and quantifying quantum correlations, or *entanglement*, in many-body quantum systems is a field of research in itself that benefits both from condensed matter and quantum information ideas. Here, we shall only discuss the von Neumann entropy as a figure of merit for entanglement. Nevertheless, there are many other measures that have been largely explored. A detailed explanation of them can be found in several reviews [3–11]. Our choice for the entropy of entanglement is based on a combination of ideas. Entropy has a clear information theory meaning. It also relates to extensive research in quantum field theory and the physics of black holes. Furthermore, its scaling properties are related to the characterization of quantum phase transitions as provided by conformal field theory. On the other hand, entanglement entropy is not a simple quantity to compute, nor a direct observable (though it relates to them).

The von Neumann entropy can be used to measure the entanglement between two parts of the system that we call A and B . Let us take a ket $|\psi\rangle_{AB}$ belonging to $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. According to the Schmidt decomposition, for any pure bipartite state we can always find two orthonormal basis $\{|\varphi_i\rangle_A\}$ and $\{|\phi_j\rangle_B\}$ such that the state $|\psi\rangle_{AB}$ can be written as

$$|\psi_{AB}\rangle = \sum_i^{\chi} \alpha_i |\varphi_i\rangle_A |\phi_i\rangle_B, \quad (2)$$

where α_i can be chosen real and positive and are called Schmidt coefficients, and $\chi \leq \min(\dim\mathcal{H}_A, \dim\mathcal{H}_B)$ is the Schmidt number. Note that the Schmidt decomposition is just the diagonalization of the matrix of coefficients in the original state which is always possible if we can perform two independent unitary transformations in A and B .

The von Neumann entropy between these bipartitions is defined as the Shannon entropy of the square of the Schmidt coefficients:

$$S_A = S_B \equiv - \sum_i \alpha_i^2 \log \alpha_i^2. \quad (3)$$

This expression can be written in terms of the reduced density matrices of each part of the system. That is,

$$S_A \equiv S(\rho_A) = -\text{tr}(\rho_A \log_2 \rho_A), \quad (4)$$

where

$$\rho_A = \text{tr}_B(|\psi\rangle_{AB}\langle\psi|_{AB}) = \sum_i \alpha_i^2 |\phi_i\rangle_B \langle\psi_i|_B. \quad (5)$$

It is easy to see that $S_A = S_B$. Thus, the surprise that A experiences when discovering its correlation with B is identical to the one of B realizing its correlation with A .

The von Neumann entropy verifies the following properties: (i) it is invariant under local unitary operations ($S_A = S_B$ is a function of α_i 's only); (ii) it is continuous (in a certain sense also in the asymptotic limit of infinite copies of the state, see e.g. [4]); (iii) it is additive: $S(|\psi\rangle \otimes |\phi\rangle) = S(|\psi\rangle) + S(|\phi\rangle)$.

In our particular case, we are going to use the entanglement entropy to study the quantum correlations of spin chains. We will be interested in determining the entanglement between a block of L contiguous spins and the rest of the chain. Then if $|GS\rangle$ represents the ground state of a system of N spins, $\rho_L = \text{tr}_{N-L}(|GS\rangle\langle GS|)$ is the reduced density matrix of the block of L contiguous spins that we will use in equation (4).

Finally, let us point out that, in the case the ground state that we study is translationally invariant, neither ρ_L nor S_L will depend on the position of the block of spins in the chain. In this case, it is easy to show that the entropy S_L is a concave function with respect to L [12]:

$$S_L \geq \frac{S_{L-M} + S_{L+M}}{2}, \quad (6)$$

where $L = 0, \dots, N$, and $M = 0, \dots, \min\{N-L, L\}$.

2.2. *XX model*

We shall now present a computation of entanglement entropy for the reduced density matrix of the ground state of the widely studied *XX* model [1, 2]. This theory captures the non-trivial structure of a quantum phase transition, while remaining simple enough to carry explicit computations throughout. The *XX* model consists of a chain of N spin- $\frac{1}{2}$ particles with nearest-neighbour interactions and an external magnetic field. Its Hamiltonian is given by

$$H_{XX} = -\frac{1}{2} \sum_{l=0}^{N-1} (\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y) + \frac{1}{2} \lambda \sum_{l=0}^{N-1} \sigma_l^z, \quad (7)$$

where l labels the N spins, λ is the magnetic field and σ_l^μ ($\mu = x, y, z$) are the Pauli matrices at site l .

Without loss of generality, we are going to consider that the magnetic field is oriented in the positive z -direction ($\lambda > 0$), since, if this was not the case, we could always map the system onto an equivalent one with $\lambda > 0$ by simply interchanging the spin states up and down.

2.3. *Ground state*

Next, we need to compute the ground state $|GS\rangle$ of the *XX* Hamiltonian (7). In order to do this, we will follow two steps: (i) first, we will perform a Jordan–Wigner transformation to rewrite H_{XX} as a quadratic form of fermionic operators, and then (ii) we will take profit of the translational invariance of the system realizing a Fourier transform which will diagonalize the Hamiltonian. A third step which is needed in the more general *XY* model, the Bogoliubov transformation, is not necessary in this particular case. Let us remark that this computation is standard and appears in many textbooks [13–15].

The Jordan–Wigner transformation maps a spin chain of interacting qubits onto an equivalent system of interacting fermions. This powerful transform is defined by the following relation between the Pauli matrices and the creation and annihilation of the fermionic modes:

$$a_l = \left(\prod_{m=0}^{l-1} \sigma_m^z \right) \frac{\sigma_l^x - i\sigma_l^y}{2}. \quad (8)$$

We, indeed, can check that the fermionic operators a_l fulfil the canonical commutation relations

$$\{a_l^\dagger, a_m\} = \delta_{lm}, \quad \{a_l, a_m\} = 0. \quad (9)$$

The idea behind the transformation is to identify the state of the spin l (0 or 1 in the computational basis) with the occupation number of the corresponding fermionic mode. Thus, in equation (8), the factor $(\sigma_l^x - i\sigma_l^y)/2$ corresponds to the operator $|0\rangle\langle 1|$ in the computational basis, and the product $\prod_{m=0}^{l-1} \sigma_m^z$ generates the appropriate sign in order to satisfy the commutation relations.

The Jordan–Wigner transformation casts the XX Hamiltonian onto

$$H_{XX} = - \sum_{l=0}^{N-1} (a_l^\dagger a_{l+1} + a_{l+1}^\dagger a_l) + \lambda \sum_{l=0}^{N-1} a_l^\dagger a_l, \quad (10)$$

which corresponds to a model of free fermions with chemical potential λ .

Now, let us exploit the translational symmetry of the system by introducing the Fourier transformed fermionic operators

$$b_k = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} a_l e^{-i\frac{2\pi}{N}kl}, \quad (11)$$

where $0 \leq k \leq N - 1$. As the Fourier transform is a unitary transformation, these new b_k operators also satisfy the canonical commutation relations and, therefore, they are fermionic operators.

The Hamiltonian, written in terms of these b_k operators, displays a diagonal structure

$$H_{XX} = \sum_{k=0}^{N-1} \Lambda_k b_k^\dagger b_k, \quad (12)$$

where the energy that penalizes (or favours, depending on the sign) the occupation of mode k is

$$\Lambda_k = \lambda - 2 \cos \frac{2\pi k}{N}. \quad (13)$$

We have assumed that the system satisfied periodic boundary conditions. If this was not the case, the Hamiltonian would not be diagonal due to an extra term proportional to $\frac{1}{N}$. In the thermodynamic limit, therefore, this extra term disappears.

We realize that, on one hand, if $\lambda > 2$, then $\Lambda_k \geq 0 \forall k$. This implies that the ground state of the system is the state annihilated by all b_k operators

$$b_k |GS\rangle = 0 \forall k, \quad (14)$$

and, therefore, it has 0 energy.

On the other hand, if $2 > \lambda \geq 0$, the ground state is the state annihilated by the operators b_k with $\Lambda_k > 0$ and b_m^\dagger with $\Lambda_m < 0$,

$$\begin{aligned} b_k |GS\rangle &= 0 & \text{if } \Lambda_k > 0 \\ b_m^\dagger |GS\rangle &= 0 & \text{if } \Lambda_m < 0, \end{aligned} \quad (15)$$

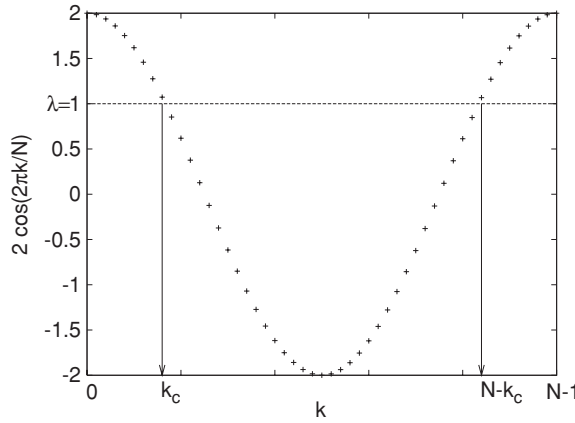


Figure 1. The two terms of Λ_k , equation (13), are plotted for the particular case $\lambda = 1$. We realize that if $2 \cos(\frac{2\pi k}{N}) > \lambda$, $\Lambda_k < 0$ while if $2 \cos(\frac{2\pi k}{N}) < \lambda$, $\Lambda_k > 0$.

and its energy is simply $\sum_m \Lambda_m \forall \Lambda_m < 0$. In figure 1 and equation (13), we can see that if $k_c \geq k \geq 0$ or $N - 1 \geq k \geq N - k_c$, where k_c is defined by

$$k_c = \left\lfloor \frac{N}{2\pi} \arccos\left(\frac{\lambda}{2}\right) \right\rfloor, \tag{16}$$

then $\Lambda_k < 0$, whereas for the rest of cases $\Lambda_k \geq 0$. In equation (16), the brackets $\lfloor \rfloor$ represent the floor function.

2.4. Entanglement entropy of a block

The strategy to get the von Neumann entropy of a block of L spins first consists in computing the correlation matrix $\langle a_m^\dagger a_n \rangle$ of the GS in this block. Then, the eigenvalues of this correlation matrix are related to the eigenvalues of the reduced density matrix of the block which are required to determine the entanglement entropy.

The simple structure of the GS, shown in equations (14) and (15), makes easy to compute its correlation matrix

$$\langle b_p^\dagger b_q \rangle = \begin{cases} \delta_{pq} & \text{if } \Lambda_p < 0 \\ 0 & \text{if } \Lambda_p > 0. \end{cases} \tag{17}$$

From now on, we will consider the case in which $2 > \lambda \geq 0$. Note that if $\lambda > 2$, then $\langle b_p^\dagger b_q \rangle = 0$ for all p and q . This case is trivial to analyse, since the correlators $\langle a_m^\dagger a_n \rangle$ are also null, and the GS is in a product state.

The next step is to go back to the Fourier transform to get the correlation matrix of the a_n operators

$$\langle a_m^\dagger a_n \rangle = \frac{2}{N} \sum_{k=0}^{k_c} \cos\left[\frac{2\pi}{N} k(m-n)\right]. \tag{18}$$

In the thermodynamic limit, the previous sum becomes an integral and it can be determined analytically. In this case, the correlation matrix of the block of L fermions in position space is

$$A_{mn} = y \langle a_m^\dagger a_n \rangle = \frac{1}{\pi} \frac{\sin k_c(m-n)}{m-n}, \tag{19}$$

where $L \geq m, n \geq 0$. Note that, by means of Wick's theorem, any operator that acts on the block can be written in terms of the correlation matrix A_{mn} . For instance,

$$\langle a_k^\dagger a_l^\dagger a_m a_n \rangle = \langle a_k^\dagger a_n \rangle \langle a_l^\dagger a_m \rangle - \langle a_k^\dagger a_m \rangle \langle a_l^\dagger a_n \rangle. \quad (20)$$

This is due to the fact that the system is Gaussian, and its eigenstates are determined by the first and second moments of some fermionic operators.

The correlation matrix A_{mn} could also be computed using the density matrix of the block ρ_L :

$$A_{mn} = \text{Tr}(a_m a_n \rho_L). \quad (21)$$

We, thus, need to invert the previous equation, that is, to compute the density matrix ρ_L from the correlation matrix A_{mn} .

The matrix A_{mn} is Hermitian, so it can be diagonalized by a unitary transformation,

$$G_{pq} = \sum_{m,n=0} u_{pm} A_{mn} u_{nq}^* = \langle g_p^\dagger g_p \rangle \delta_{pq}, \quad (22)$$

where $g_p = \sum_m u_{pm} a_m$. In this case, the density matrix of the block must also verify

$$G_{mn} = \text{Tr}(g_m^\dagger g_n \rho_L) = v_m \delta_{mn}, \quad (23)$$

which implies that ρ_L is uncorrelated and it can be written as

$$\rho_L = \varrho_1 \otimes \cdots \otimes \varrho_L, \quad (24)$$

where ϱ_m is the density matrix corresponding to the m th fermionic mode excited by g_m^\dagger .

In order to determine the eigenvalues of the density matrix of one mode, let us express the g_m, g_m^\dagger and ϱ_m operators in its matrix representation. That is,

$$g_m = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad g_m^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (25)$$

and

$$\varrho_m = \begin{pmatrix} \alpha_m & \beta_m \\ \beta_m^* & 1 - \alpha_m \end{pmatrix}, \quad (26)$$

where α_m and β_m are the matrix elements of ϱ_m that we want to determine. It is easy to see that $\beta_m = 0$, since

$$\langle g_m \rangle = \text{Tr}(g_m \rho_L) = \beta_m = 0. \quad (27)$$

Moreover, rewriting equation (23) in terms of these matrices

$$\text{Tr}(g_m^\dagger g_m \rho_L) = \text{Tr} \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_m & 0 \\ 0 & 1 - \alpha_m \end{pmatrix} \right] = v_m \quad (28)$$

we realize that $\alpha_m = v_m$.

The entanglement entropy between the block and the rest of the system is, therefore,

$$S_L = \sum_{l=1}^L H_2(v_l), \quad (29)$$

where $H_2(x) = -x \log x - (1-x) \log(1-x)$ denotes the binary entropy.

Summing up, the three steps that we have to follow in order to compute the entanglement entropy of the GS of a block of L spins for the XX model are (i) to determine the correlation matrix A_{mn} by evaluating equation (19) for $L \geq m, n \geq 0$, (ii) to diagonalize this correlation matrix and, with its eigenvalues, (iii) to compute the entanglement entropy according to equation (29).

Let us emphasize that this method is computationally efficient, since its computational cost scales polynomially with the number of spins of the block $O(L^3)$, whereas the Hilbert space of the problem has dimension 2^N .

It is also necessary to recall a quite subtle point that we have skipped along our previous discussion. It turns out that there is no need to perform a final transformation back to spins, that is, there is no need to invert the Jordan–Wigner transformation. This is due to the fact that the coefficients of a given state are identical when written in terms of the spin basis or in terms of the fermionic a_l operators. More precisely,

$$\begin{aligned} |\psi\rangle &= \sum_{i_1, i_2, \dots, i_n} C^{i_1, i_2, \dots, i_n} |i_1, i_2, \dots, i_n\rangle \\ &= \sum_{i_1, i_2, \dots, i_n} C^{i_1, i_2, \dots, i_n} (a_1^\dagger)^{i_1} (a_2^\dagger)^{i_2} \dots (a_n^\dagger)^{i_n} |\text{vac}\rangle. \end{aligned} \quad (30)$$

Thus, the same coefficients appear in the ket, either when written in the initial spin basis, or when expressed as creation operators in the fermionic vacuum, $|\text{vac}\rangle$. Consequently, the reduced density matrix entropy of entanglement is identical for both expressions.

Finally, let us mention that the computation of the geometric entropy of Gaussian systems has been systematized in [16]. In particular, it is shown that for solvable fermionic and bosonic lattice systems, the reduced density matrices can be determined from the properties of the correlation functions. This subject is reviewed in [17].

2.5. Scaling of the entropy

It is now easy to compute the entanglement entropy of the ground state of the XX model for arbitrary values of the block size L and magnetic field λ .

In figure 2, we show how the entropy of the reduced density matrix of a block of L spins varies with L for different values of the magnetic field. The maximum entropy is reached for $\lambda = 0$. In particular, we recover the result in [2] and see that for $\lambda = 0$ the leading scaling of the entropy is perfectly fitted by a logarithm,

$$S_L = \frac{1}{3} \log_2 L + a, \quad (31)$$

where a is a constant that was determined analytically in [18].

As we increase the magnetic field, but it is still less than 2, the entropy decreases although it keeps its logarithmic behaviour with L . When $\lambda > 2$, the entropy saturates to zero, since the ground state is already in a product state in the spin basis corresponding to the ferromagnetic phase, $\prod_i |\uparrow\rangle_i$.

The relation between logarithmic scaling and entropy is confirmed by similar computations in different models. The general result is that entanglement entropy obeys a logarithmic scaling relation at critical points, that is when the system is at a phase transition, whereas a saturation of entanglement is found away from criticality. This universal logarithmic at critical points must emerge from the basic symmetry that characterizes phase transitions, namely conformal invariance. We shall come to this developments in the next section.

Let us mention that the scaling of entanglement entropy was formerly studied in the context of quantum field theory and black-hole physics. There, the system sits in higher dimensions. The entanglement entropy scales following an area law that we shall discuss later on. Yet, it is important to note that one-dimensional systems are an exception to the area law. Entanglement pervades the system at any distance, not staying just at the point-like borders of a block.

Summing up, we have seen that the scaling entropy is a good witness for quantum phase transitions. Many other studies of different measures of entanglement at quantum phase

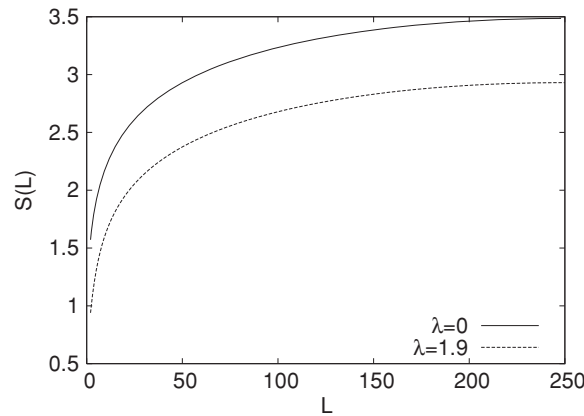


Figure 2. Entropy of the reduced density matrix of L spins for the XX model in the limit $N \rightarrow \infty$, for two different values of the external magnetic field λ . The maximum entropy is reached when there is no applied external field ($\lambda = 0$). The entropy decreases while the magnetic field increases until $\lambda = 2$ when the system reaches the ferromagnetic limit and the ground state is a product state in the spin basis.

transitions have been presented recently. Let us here mention that in [19, 20], quantum phase transitions are characterized in terms of the overlap (fidelity) function between two ground states obtained for two close values of external parameters. When crossing the critical point, a peak of the fidelity is observed.

2.6. Entanglement entropy and Toeplitz determinant

Before finishing this section, we would like to sketch how the particular structure of the correlation matrix in equation (19) allows us to derive an analytical expression for the scaling law of the entanglement entropy. This result is presented in [18].

In order to obtain an analytical expression for the entanglement entropy, let us introduce the function

$$D_L(\mu) = \det(\tilde{A}(\mu)), \tag{32}$$

where $\tilde{A}(\mu) \equiv \mu I_L - A$, I_L is the identity matrix of dimension L , and A is the correlation matrix defined in equation (19). If we express the matrix A in its diagonal form, we trivially have

$$D_L(\mu) = \prod_{m=1}^L (\mu - v_m), \tag{33}$$

where v_m , with $m = 1, \dots, L$, are the eigenvalues of A . Then according to the Cauchy residue theorem, the entanglement entropy S_L can be expressed in terms of an integral in the μ -complex plane as follows:

$$\begin{aligned} S_L &= \lim_{\epsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \frac{1}{2\pi i} \oint_{C(\epsilon, \delta)} e(1 + \epsilon, \mu) d \ln D_L(\mu) \\ &= \sum_{m=1}^L H_2(v_m), \end{aligned} \tag{34}$$

where $c(\epsilon, \delta)$ is a closed path that encircles all zeros of $D_L(\mu)$ and $e(1 + \epsilon, \mu)$ is an arbitrary function that is analytic in the contour $c(\epsilon, \delta)$ and verifies $e(1, v_m) = H_2(v_m) \forall m$.

Thus, if we could obtain an analytical expression for the $D_L(\mu)$ function, we would be able to get a closed analytical result for the entanglement entropy.

Note that both the correlation matrices A_{mm} , defined in equation (19), and \tilde{A} are Toeplitz matrices, that is to say, matrices in which each descending diagonal from left to right is constant,

$$A = \begin{pmatrix} f_0 & f_{-1} & \dots & f_{1-L} \\ f_1 & f_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ f_{L-1} & \dots & \dots & f_0 \end{pmatrix}, \tag{35}$$

where, in this case, $f_m = \frac{1}{\pi} \frac{\sin k_c m}{m}$.

The asymptotic behaviour (when $L \rightarrow \infty$) of the determinant of Toeplitz matrices has been widely studied in many cases, giving the famous Fisher–Hartwig conjecture (see [21–25]). In our particular case, the expression for the determinant of \tilde{A} was proven in [23] and, therefore, it is a theorem instead of a conjecture. In this way, we may insert this result in equation (34), perform the corresponding complex integral and obtain the asymptotic analytical expression for the entanglement entropy of the XX model. This computation is presented explicitly in [18] with the final result

$$S_L = \frac{1}{3} \ln L + \frac{1}{6} \ln \left(1 - \left(\frac{\lambda}{2} \right)^2 \right) + \frac{\ln 2}{3} + \Upsilon_1, \tag{36}$$

where

$$\Upsilon_1 = - \int_0^\infty dt \left\{ \frac{e^{-t}}{3t} + \frac{1}{t \sinh^2(t/2)} - \frac{\cosh(t/2)}{2 \sinh^3(t/2)} \right\}. \tag{37}$$

Indeed, we realize that this analytical expression for the scaling of the entanglement entropy is compatible with the numerical fit of equation (31) and, moreover, it fixes the value of the independent term.

This procedure is also used to obtain an analytical expression for the entanglement entropy of the XY model in [26]. In [27], the scaling of the Renyi entropy is determined for the XY model in terms of Klein’s elliptic λ —a function showing a perfect agreement with the previous results in the particular case in which the Renyi entropy becomes the von Neumann entropy.

3. Scaling of entanglement

The logarithmic scaling law that entanglement entropy obeys in the critical regime is a sign of the conformal symmetry of the system. For second-order phase transitions, the correlation length diverges and the system becomes scale invariant. This scaling symmetry gets enlarged to the conformal group [28] which, in the case of one-dimensional systems, allows for a very precise characterization of the operator structure of the underlying theory. The development of conformal field theory is a remarkable achievement that we cannot present in this short review [29, 30].

3.1. One-dimensional systems

For a conformal theory in 1+1 dimensions, the scaling behaviour of the entropy was proven to be logarithmic in [31]. The general result reads

$$S_L \sim \frac{c + \bar{c}}{6} \log_2 L, \quad (38)$$

where c and \bar{c} are the so-called central charges for the holomorphic and anti-holomorphic sectors of the conformal field theory. These central charges classify conformal field theories and are *universal* quantities which depend only on the basic properties of the system, such as effective degrees of freedom of the theory, symmetries or spatial dimensions, and they are independent of the microscopic details of the model. For free bosons $c = 1$, whereas for free fermions $c = 1/2$.

This result matches perfectly our geometric entropy computation of the critical XX model. In this case, the central charge $c = \bar{c} = 1$ and the model is seen to belong to the free boson universality class.

The previous result of equation (38) was further elaborated and extended to finite systems, finite temperature and disjoint regions in [31–33]. For instance, the scaling of the entropy for a system under periodic boundary conditions reads

$$S_A \sim \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi \ell}{L} \right) + c'_1, \quad (39)$$

whereas for the open boundary conditions the case is

$$S_A = \frac{c}{6} \log \left(\frac{2L}{\pi a} \sin \frac{\pi \ell}{L} \right) + \tilde{c}'_1. \quad (40)$$

In [34], the scaling of the entropy of a conformal semi-infinite chain is presented. In [35], conformal symmetry is further exploited and an analytical computation of the distribution of eigenvalues of the reduced density matrix of a block in a one-dimensional systems is presented.

Let us finally mention that the scaling of entanglement has been also studied for other entanglement measures by means of conformal field theory. In particular, in [36, 37], it is shown that the single copy entanglement scales as

$$E_1(\rho_L) = \frac{c}{6} \log L - \frac{c}{6} \frac{\pi^2}{\log L} + O(1/L). \quad (41)$$

Note that entropy sub-leading corrections to scaling are suppressed as $1/L$ whereas single copy entanglement suffers from $1/\log L$ modifications. This makes the numerical approach to the latter more difficult.

3.2. Conformal field theory and central charge

We stated above that the central charge is a quantity that characterizes the universality class of a quantum phase transition. We also mentioned in the introduction that a possible figure of merit for entanglement could be constructed from correlation functions made out of an operator which is always present in any theory. Let us now see that both ideas merge naturally.

In 1+1 dimensions, conformal field theories are classified by the representations of the conformal group [29]. The operators of the theory fall into a structure of highest weight operators and its descendants. Each highest weight operator carries some specific scaling dimensions which dictate those of its descendants. The operators close an algebra implemented into the operator product expansion. One operator is particularly important: the energy–momentum tensor $T_{\mu\nu}$, which is a descendant of the identity. It is convenient to introduce

holomorphic and anti-holomorphic indices defined by the combinations $T = T_{zz}$ and $\bar{T} = T_{\bar{z}\bar{z}}$ where $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$. Denoting by $|0\rangle$ the vacuum state, the central charge of a conformal field theory is associated with the coefficient of the correlator

$$\langle 0|T(z)T(0)|0\rangle = \frac{c}{2z^4}, \quad (42)$$

and the analogous result for \bar{c} in terms of the correlator $\langle 0|\bar{T}(z)\bar{T}(0)|0\rangle$. A conformal field theory is characterized by its central charge, the scaling dimensions and the coefficients of the operator product expansion. Furthermore, unitary theories with $c < 1$ only exist for discrete values of c and are called minimal models. The lowest lying theory corresponds to $c = \frac{1}{2}$ and represents the universality class of a free fermion.

The central charge plays many roles in a conformal field theory. It was introduced above as the coefficient of a correlator of energy–momentum tensors, which means that it is an observable. The central charge also characterizes the response of a theory to a modification of the background metric where it is defined. Specifically, the scale anomaly associated with the lack of scale invariance produced by a non-trivial background metric is

$$\langle 0|T_\mu^\mu|0\rangle = -\frac{c}{12}R, \quad (43)$$

where R is the curvature of the background metric. This anomaly can also be seen as the emergence of a non-local effective action when the field theory modes are integrated out in a curved background.

Therefore, the central charge which appears as the coefficient of the entanglement entropy is naturally related to the stress tensor, which is the operator that is guaranteed to exist in any theory. It is also possible to derive a direct relation between entropy and the trace of the stress tensor as shown in the original [31].

3.3. Area law

The conformal group does not constrain the structure of the Hilbert space in spatial dimensions higher than 1 as much as it does in one dimension. Actually, the conformal group no longer brings an infinite number of conserved charges (as it does in one dimension) but becomes a finite group.

Nevertheless, a geometric argument establishes the scaling behaviour of entropy. The basic idea goes as follows. Let us consider a volume of spins (or any local degrees of freedom) contained in a larger space. For theories with local interactions, it is expected that entanglement will be created between the degrees of freedom that lie outside and inside the surface that encloses the volume we are considering. It follows that the entropy should naturally scale as an area law even if the model displays a finite correlation length.

These arguments were put forward in the study of entanglement entropy in quantum field theory as a possible source for black-hole entropy [38–40]. Furthermore, the relation between the entropy and the effective action in a curved background was developed in [41]. Let us mention these results. For general quantum field theories the entropy is a divergent extensive quantity in more than one spatial dimension obeying an area law

$$S_L \sim c_1 \left(\frac{L}{\epsilon}\right)^{d-1}, \quad d > 1, \quad (44)$$

where L^d is the size of the volume, ϵ stands for an ultraviolet regulator and the coefficient c_1 counts components of the field which is considered. This coefficient is again found in the

effective action on a gravitational field and, thus, in the trace anomaly as a divergent term. A form for the former can be found as

$$\Gamma_{\text{eff}} = \int_{s_0}^{\infty} ds \frac{e^{-sm^2}}{s^{d/2}} \left(\frac{c_0}{s} + c_1 R + c_{2F} F + c_{2G} G + \dots \right), \quad (45)$$

where s_0 acts as an ultraviolet regulator, R is the curvature, F the Weyl tensor and G the Euler density. The main conceptual result to be retained is that entropy measures a very basic counting of degrees of freedom. Note that previous efforts to make a general c-theorem are all based on c_{2F} and c_{2G} , not on c_1 . In one spatial dimension, the effective action has a unique structure proportional to the central charge. That is, the central charge takes over all manifestations of the trace anomaly, at variance with the separate roles that appear in higher dimensions.

It is worth mentioning that computations done in massive theories in any number of dimensions show that $S_L(m \neq 0) - S_L(m = 0)$ comes out to be ultraviolet finite [42]. Actually, the ultraviolet cut-off cancels in the computation. This is precisely what is needed to make the RG flow meaningful in such a case. This comment hints at the non-trivial issue about observability of the entropy. The standard prejudice is that the leading area law coefficient is not observable since it comes divided by a necessary ultraviolet cut-off. Yet, if such a coefficient is also responsible for finite corrections, the situation may not be as trivial.

A review on methods to calculate the entanglement entropy for free fields and some particular examples in two, three and more dimensions is presented in [43]. Further explicit computations of area law scaling of entropy in spin and harmonic systems in higher dimensions can be found in [44–47]. A quite remarkable result found in [48] is the fact that certain fermionic systems may develop logarithmic violations of the area law, while keeping local interactions. This is somehow analogous of the logarithmic scaling in one dimension. The system is more correlated than what is expected from pure geometrical arguments. In this respect, the leading term in the scaling of the entropy for fermionic systems was computed analytically assuming the Widom conjecture in [49]. This result was checked numerically for two critical fermionic models in [46] which found a good agreement.

For other steps into a description of systems with two spatial dimensions in the framework of conformal field theory, see [50, 51]. For a class of critical models in two spatial dimensions (including the quantum dimer model), it is found that $S(\rho_I) = 2f_s(L/a) + cg \log(L/a) + O(1)$, where L is the length of the boundary area, f_s is an area law coefficient that is interpreted as a *boundary free energy* and g is a coefficient that depends on the geometric properties of the partition. That is, in addition to a non-universal area law, one finds a universal logarithmically divergent correction. For a further discussion of steps towards a full theory of entanglement entropies in $(d + 1)$ -dimensional conformal field theories, see [50, 51].

A particularly interesting issue is the holographic entanglement entropy that emerges from the AdS (anti-de Sitter)/CFT correspondence. The AdS/CFT correspondence is the conjectured equivalence between a quantum gravity theory defined on one space, and a quantum field theory without gravity defined on the conformal boundary of this space. The entanglement entropy of a region of the boundary in the conformal field theory is then related to the degrees of freedom of part of the AdS space in the dual gravity side. In [51, 52], this relation is established explicitly and, in [53], the recent progress on this topic is presented.

Let us also mention the line of research that deals with topological entropy. Some Hamiltonians produce states such that a combination of geometric entropies exactly cancels the dominant area law. Then a topological entropy term characterizes the system [54]. This subject is nowadays a large field of research that we cannot include in the present paper. In

this respect, a review on the scaling of the entanglement entropy of 2D quantum systems in a state with topological order is presented in [55].

4. Other models

We can find in the literature the computation of the scaling of the entanglement entropy for other spin models. In XY and XXZ models, this logarithmic scaling will confirm the role of the underlying conformal symmetry. We shall also discuss that in disordered systems, although the conformal symmetry is lost for one particular realization of the disorder, we recover the logarithmic scaling of the entropy with a different central charge of the corresponding homogeneous model, if we calculate the average over all the possible realizations of the disorder. We shall also study the scaling of entropy in systems where the notion of geometric distance is lost. This is the case of the Lipkin–Meshkov–Glick model, in which the logarithmic behaviour of the entropy is due to the equilibrium of a competition between the long-range interactions, that try to increase the entanglement, and the symmetries of the problem, that force the ground state to belong to a reduced subspace of the Hilbert space. A different case is those systems composed of itinerant particles. In particular, we will present the scaling of entropy of the Laughlin wavefunction.

4.1. The XY model

The XY model is defined as the XX model in equation (7), adding an extra parameter γ that determines the degree of anisotropy of the spin–spin interaction in the XY plane. Its Hamiltonian reads

$$H_{XY} = -\frac{1}{2} \sum_l \left(\frac{1+\gamma}{2} \sigma_l^x \sigma_{l+1}^x + \frac{1-\gamma}{2} \sigma_l^y \sigma_{l+1}^y + \lambda \sigma_l^z \right), \quad (46)$$

where, as in the previous section, l labels the N spins, σ_l^μ ($\mu = x, y, z$) are the Pauli matrices and λ is the transverse magnetic field in the z -direction. This notation will be also followed for the rest of models that are going to be presented.

Note that if $\gamma = 0$, we recover the XX model, whereas if $\gamma = 1$, it becomes the quantum Ising model with a transverse magnetic field, with Hamiltonian

$$H_{\text{Ising}} = -\frac{1}{2} \sum_l (\sigma_l^x \sigma_{l+1}^x + \lambda \sigma_l^z). \quad (47)$$

The XY model was solved in detail in [2]. In order to do this, the previous works on spin chains required to solve the XY Hamiltonian were reviewed. In concrete, the XY model without magnetic field was solved exactly in [56], the spectrum of the XY model with a magnetic field was computed in [57], the correlation function for this model was obtained in [58] and the entropy S_L was computed in [1].

Later, an extent analytical analysis of the entropy of the XY spin chain was presented in [26]. In this work, in a similar way as we have seen previously for the XX model, an analytical expression for the scaling of the entanglement entropy is determined for the XY model by means of Toeplitz determinants.

The XY model with $\gamma \neq 0$ is critical for $\lambda = 1$. In this case, the entropy of a block scales as

$$S_{XY}(L) = \frac{1}{6} \log_2 L + a(\gamma), \quad (48)$$

where $a(\gamma)$ is a function that only depends on γ . This entanglement behaviour corresponds to the scaling dictated by a conformal theory, equation (38), with central charge $c = 1/2$. The XY model, therefore, falls into the free fermion universality class.

In the non-critical case, that is for $\lambda \neq 1$, the entanglement entropy saturates to a constant.

Let us mention that an exact relationship between the entropies of the XY model and the XX model has been found recently [59]. Using this relation it is possible to translate known results between the two models and obtain, among others, the additive constant of the entropy of the critical homogeneous quantum Ising chain and the effective central charge of the random XY chain.

Finally, with respect to the particular case of the Ising model in [60], the computation of the leading correction to the bipartite entanglement entropy at large sub-system size, in integrable quantum field theories with diagonal scattering matrices, is presented. This result is used to compute the exact value of the saturation in the Ising model and is in good agreement with numerical results. This work is reviewed in detail in [61].

4.2. The XXZ model

The XXZ model consists of a chain of N spins with nearest-neighbour interactions and an external magnetic field. Its Hamiltonian is given by

$$H_{XXZ} = \sum_l \left(\frac{1}{2} [\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y + \Delta \sigma_l^z \sigma_{l+1}^z] + \lambda \sigma_l^z \right), \quad (49)$$

where Δ is a parameter that controls the anisotropy in the z -direction.

As happened for the γ parameter of the XY model, the Δ parameter of the XXZ model has two particular interesting values. If $\Delta = 0$, we trivially recover the XX model, and if $\Delta = 1$, the system becomes the XXX model that has a fully isotropic interaction

$$H_{XXX} = \sum_l \left(\frac{1}{2} [\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y + \sigma_l^z \sigma_{l+1}^z] + \lambda \sigma_l^z \right). \quad (50)$$

The XXZ model can be solved analytically by means of the Bethe ansatz technique [62]. Bethe ansatz takes profit of the two symmetries of the system to find its eigenstates. The first symmetry is the rotational symmetry with respect to the z axis. It implies that the z -component of the total spin, $S_z = 1/2 \sum_l \sigma_l^z$, must be conserved and, therefore, the Hamiltonian must be diagonal in boxes of constant S_z . The other symmetry is the translational invariance that allows us to diagonalize these boxes using a kind of generalized Fourier transform. Once the ground state is obtained, the correlation functions can be computed in terms of certain determinants (see [63]). This model is qualitatively different from the XY model, since it presents a point of non-analyticity of the ground-state energy for finite systems. In the XY model, the level crossing between the ground state and the first excited state only occurs in the thermodynamic limit. In this case, instead, the terms of the Hamiltonian $\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y$, $\sigma_l^z \sigma_{l+1}^z$ and σ_l^z commute and are independent of Δ and λ , which implies that there will be an actual level crossing.

Both the isotropic case and the anisotropic one for $\lambda = 0$ are solved in [2, 64]. The phases of the system are found to be as follows.

- In the XXX model, equation (50), there are two limit behaviours. On one hand, when $|\lambda| > 2$, the system is gapped and it is in a product state in which all spins point to the direction of the magnetic field (ferromagnetic phase). On the other hand, for $\lambda = 0$ the magnetization is zero and the system is in an entangled state (anti-ferromagnetic case). In the interval between these two cases $2 > \lambda \geq 0$ the system is gap less and, therefore, critical.

- With respect to the anisotropic case with a magnetic field equal to zero, the system shows a gap-less phase in the $1 \geq \Delta \geq -1$ interval. Outside this interval, there is a gap between the ground and the first excited states. These two phases are separated by two phase transitions in $\Delta = 1$ and $\Delta = -1$. The first one is a Kosterlitz–Thouless phase transition, while the second one is of first order.

The scaling of the entanglement entropy is presented in [2, 65]. These numerical results show that the entanglement entropy behaviour converges to a logarithmic scaling as the size of the system increases, if the system is critical. In contrast, if the system is not in a critical phase, the entropy saturates to a constant value. In particular, in the isotropic model without magnetic field, the entropy scales as

$$S_L \sim \frac{1}{3} \log_2 L, \quad (51)$$

which means that the XXX model with $\lambda = 0$ has central charge $c = 1$ and falls into the universality class of a free boson.

Finally, let us mention that, recently, analytic expressions for reduced density matrices, several correlation functions and the entanglement entropy of small blocks (up to six spins) have been found for the XXZ model with $\Delta = 1/2$ (see [66] and [67]).

4.3. Disordered models

So far, we have only considered translational invariant systems. This symmetry plus the scaling invariance at a critical point produces the conformal symmetry that implies the universal properties of the scaling of entanglement. Nevertheless, natural systems exhibit a certain amount of *disorder* due to impurities and imperfections of the system. This disorder breaks the translational symmetry and we wonder what happens with the scaling of the entropy taking into account that the conformal invariance is lost.

This question was addressed in [68]. They computed analytically the block entropy for the Heisenberg, XX and quantum.

Ising models with random nearest-neighbour couplings under the hypothesis of strong disorder by means of the real space renormalization group technique. This approach was introduced in [69] and was generalized in [70].

This strong disorder hypothesis assumes that if one takes the strongest coupling of the chain, its neighbours are much weaker than it. Thus, it is possible to diagonalize this strongest bond independently of the rest of the system, project the system onto the ground state of this subspace (a singlet for the previous models) and then perform perturbation theory with respect to the neighbour couplings. The final result is that two sites have been eliminated and the Hamiltonian energy scale has been reduced. This process can be iterated until we arrive at the ground state of the system which is a random singlet phase, that is to say, a set of singlets connected randomly and for arbitrarily long distances.

Note that although this method is not correct when applied to a system with weak disorder, it becomes asymptotically correct at large distances [69].

For a particular realization of the disorder, the translational symmetry of the system is broken and, therefore, the conformal symmetry too. Hence, the scaling of the entanglement entropy of this realization of the disorder will not be logarithmic, but fluctuating.

In [68], it was shown that although the conformal symmetry is broken, if we take the average over all the realizations of the disorder, the entropy keeps scaling logarithmically with an effective central charge $\tilde{c} = c \ln 2$, where c is the central charge for the same model but without disorder. This result has been further checked numerically both for the XX model in [71] and for the Heisenberg model in [72].

In [71], the disordered XX spin- $\frac{1}{2}$ chain under periodic boundary conditions and positive random spin couplings chosen in a flat uniform distribution within the interval $[0, 1]$ was studied. The magnetic field was set to zero. It was shown that for a block large enough (larger than 20 spins), the entropy scales logarithmically according to [68], using around 10^4 samples for $N = 500, 1000, 2000$ and 2×10^4 samples for $100 \leq N \leq 400$, in order to do average over all the possible realizations of the disorder.

The same result was shown for the Heisenberg model in [72]. In this work, a uniform distribution in the interval $[0, 1]$ for the couplings between the spins was also chosen. For a system of $N = 50$ and after averaging the entanglement entropy over 10^4 different configurations of disorder, the logarithmic scaling of the entropy with an effective central charge $\tilde{c} = c \ln 2$ is recovered. Let us point out that these one-dimensional systems are particular cases of chains of quantum group (or q -deformed) spins studied in [73]. It is also interesting to mention that this robustness of the entanglement scaling with respect to the disorder is not kept for other models such as the Bose–Hubbard model (see [74]).

In the case of higher dimensions, the scaling of the entanglement entropy in the 2D random Ising model was studied in [75, 76]. In particular, in [76], the entanglement entropy of an $L \times L$ region located at the centre of a square lattice which is governed by the Hamiltonian

$$H = - \sum_{\langle i,j \rangle} J_{ij} \sigma_i^z \sigma_j^z - \sum_i \lambda_i \sigma_i^x \tag{52}$$

was computed. The Ising couplings J_{ij} and the transverse magnetic fields λ_i take random values given by the uniform probability distributions in the intervals $[0, 1]$ and $[0, \lambda_0]$, respectively. By means of a generalized version for two dimensions of the real space renormalization group, it was found that the critical field is at $\lambda_0^c = 5.37 \pm 0.03$, and for both critical and non-critical λ_0 the entropy scaling fulfils the area law: $S(L) \sim L$ is the leading term.

Let us mention that some disordered spin systems have also been studied from the fidelity susceptibility point of view in [77]. Finally, it is interesting to point out that, in other systems, the translational invariance is not broken by means of random couplings but due to a quantum impurity or a physical boundary. The behaviour of the entanglement entropy in this kind of systems is reviewed in [78].

4.4. The Lipkin–Meshkov–Glick model

The Lipkin–Meshkov–Glick (LMG) model was proposed in [79]. Unlike the previous models we have considered, where the spins had short-range interactions, in the LMG model, each spin interacts with all the spins of the system with the same coupling strength. This system is described by the Hamiltonian

$$H_{\text{LMG}} = -\frac{1}{N} \sum_{i < j} (\sigma_i^x \sigma_j^x + \gamma \sigma_i^y \sigma_j^y) - \lambda \sum_i \sigma_i^z. \tag{53}$$

Note the apparent similarity between this model and the XY model in equation (46). The essential difference between them is that while in the XY Hamiltonian the interaction only takes place between nearest neighbours, in the LMG model all spins interact among themselves. This highly symmetric interaction pattern forces the loss of the notion of geometry, since there is no distance between the spins. This implies that it no longer makes sense to define a block of L spins as a set of L contiguous spins or to study decays of the correlations between two spins.

As in previous cases, our aim is to study the scaling properties of the entanglement entropy for the ground-state reduced density matrix of a block of L spins with respect to the rest of

N_L spins. We face a somewhat contradictory situation. On one hand, we expect that the non-local connectivity of the interactions would produce a ground state more entangled than that emerges from nearest-neighbour interaction models. On the other hand, the symmetry of the Hamiltonian implies that all the spins must be indistinguishable in the ground state; therefore, it must belong to a symmetric subspace, which restricts its entanglement. The explicit computation will clarify this issue.

The Hamiltonian (53) can be written in terms of the total spin operators $S_\alpha = \sum_i \sigma_\alpha^i / 2$ as

$$H = -\frac{1}{N}(1 + \gamma)(\mathbf{S}^2 - S_z^2 - N/2) - 2\lambda S_z - \frac{1}{2N}(1 - \gamma)(S_+^2 + S_-^2), \quad (54)$$

where S_\pm are the ladder angular momentum operators. In equation (54), we realize that $[\mathbf{S}^2, H] = 0$ and, therefore, we can diagonalize the Hamiltonian in boxes of constant S . From equation (54), it is easy to see that the ground state must belong to the subspace of $S = N/2$. Then we have to span this subspace in terms of a basis $|N/2, M\rangle$ fully symmetric under the permutation group and eigenstates of \mathbf{S}^2 and S_z . These states $|N/2, M\rangle$ are called Dicke states.

Note that the restricted subspace where the ground state must live due to the symmetries of the Hamiltonian will limit the scaling of the entanglement entropy of a block of L spins with respect to the remaining N_L . As the ground-state reduced density matrix is spanned by the set of $(L+1)$ Dicke states, the entropy of entanglement obeys the constrain $S_{L,N} \leq \log_2(L+1)$ for all L and N , where the upper bound corresponds to the entropy of the maximally mixed state $\rho_{L,N} = \mathbb{1}/(L+1)$ in the Dicke basis. This argument implies that entanglement, as measured by the von Neumann entropy, cannot grow faster than the typical logarithmic scaling observed in the previous cases.

Both the ground state and the entanglement entropy were computed for the LMG model in [80]. For the isotropic case ($\gamma = 1$) and in the thermodynamic limit ($N, L \gg 1$), H_{LMG} is diagonal in the Dicke basis. Then, for $\lambda \geq 1$, the entanglement entropy is strictly zero since the ground state is in a fully polarized product state. Instead, if $1 > \lambda \geq 0$, we recover the logarithmic scaling of the entropy:

$$S_{L,N}(\lambda, \gamma = 1) \sim \frac{1}{2} \log [L(N - L)/N]. \quad (55)$$

Although the kind of scaling does not depend on the strength of the magnetic field, its absolute value is smaller for weaker magnetic fields, according to the equation

$$S_{L,N}(\lambda, \gamma = 1) - S_{L,N}(\lambda = 0, \gamma = 1) \sim \frac{1}{2} \log (1 - \lambda^2), \quad (56)$$

and thus diverges, at fixed L and N , in the limit $h \rightarrow 1^-$.

In the anisotropic case, we can study the limits of very strong and very weak magnetic fields. On one hand, when $\lambda \rightarrow \infty$, the GS is in the product state $\prod_i |\uparrow\rangle_i$ and therefore is not entangled. In the thermodynamic limit, this state is also the ground state just for $\lambda > 1$. On the other hand, for $\lambda \rightarrow 0$ the entanglement entropy saturates and goes to a constant that depends on γ . In the particular case of $\gamma = 0$, the ground state is degenerate and lives in the subspace generated by the states $\prod_i |\rightarrow\rangle_i$ and $\prod_i |\leftarrow\rangle_i$, where $|\rightarrow\rangle$ and $|\leftarrow\rangle$ are the eigenstates of the σ^x operator. In practice, this degeneration would be broken by any perturbation of the environment.

These two different phases suggest the existence of a quantum phase transition between $\lambda \gg 1$ and $\lambda \ll 1$. In particular, it has, numerically, been checked in [80] that, in the thermodynamic limit, the entanglement entropy displays a logarithmic divergence around $\lambda_c = 1$ according to the law

$$S_{L,N}(\lambda, \gamma) \sim -\log |1 - \lambda|. \quad (57)$$

Indeed, it is shown that at $\lambda = 1$ the entropy scales logarithmically with a coefficient that depends on γ . However, in the thermodynamic limit, this coefficient is independent of γ and takes a value closed to $1/3$. In [81], the previous relation, equation (57), is computed analytically obtaining the same result and fixing the coefficient to $1/3$. In the same work, the finite size corrections to the scaling of entropy are also studied.

Although the behaviour of entanglement is very similar to the XY model, that is to say, it scales logarithmically in the critical point and saturates to a constant in the non-critical phase, the reasons of these scaling laws are different. In the XY model, entanglement is limited by the facts that interactions are local and the system is translationally invariant. At the critical point, the correlation length becomes infinite, the system is conformal symmetric and the logarithmic scaling of the entanglement entropy appears as a manifestation of this symmetry. Instead, in the LMG model, the long-range interactions should allow for larger correlations, that is, larger entanglement. Nevertheless, the symmetries of the system restrict the subspace where the GS must belong and, therefore, the scaling law of entanglement. The final result is the same logarithmic scaling law but which has nothing to do with any underlying conformal symmetry.

Finally, let us mention that other analytical calculations of the spectrum of the LMG model in both the thermodynamic limit and finite size case have appeared recently [82]. Moreover, the entanglement entropy for general free bosonic two-mode models is presented in [83]. In particular, a complete classification of the possible scaling behaviours for the entanglement entropy in the related collective models such as the LMG model, the Dicke model or the Lieb–Mattis model is obtained.

4.5. Particle entanglement

In a similar way than the LMG model, where the notion of distance was lost, one can try to compute the entanglement entropy in systems of moving fermions and bosons. In such itinerant systems, as the particles are indistinguishable, moving and partially delocalized, it is not obvious to define the geometric entropy.

What we, indeed, can compute is the von Neumann entropy for any subset of particles for a system of N indistinguishable particles in the state $\Psi(r_1, \dots, r_n)$. Note that, in this case, this von Neumann entropy cannot be interpreted as the number of distillable EPR pairs. Due to the symmetrization, it is impossible to associate a label with the particles and perform the appropriate distillation operations. This is a subtle difference with respect to the LMG model.

A particular interesting physical system is the fractional quantum Hall effect (FQHE) [84]. Although a complete understanding of it is still missing, it is commonly believed that the interactions between the particles are essentially responsible for the strange states of matter that the 2D electron gas shows at some particular values of the transverse magnetic field. These states would present a new kind of order called topological order and their quasi-particle excitations are neither bosons nor fermions, but anyons, that is to say, quasi-particles with any-statistics [85]. In this respect, in 1983, Laughlin proposed an ansatz for the wavefunction of the ground state of the system [86]. This wavefunction is defined by

$$\Psi_L^{(m)}(z_1, \dots, z_n) \sim \prod_{i < j} (z_i - z_j)^m e^{-\sum_{i=1}^n |z_i|^2/2}, \quad (58)$$

where $z_j = x_j + iy_j$, $j = 1, \dots, n$, stands for the position of the j th particle. It describes fractional quantum Hall state at a filling fraction $\nu = 1/m$, where m is an integer number.

In particular, in [87], the entanglement entropy of k particles with respect to the rest of the system is computed for the Laughlin wavefunction with filling fraction one:

$$S_{n,k} = \log_2 \binom{n}{k}. \quad (59)$$

Note that, in this case, although the state also belongs to a completely anti-symmetric subspace, the entanglement entropy of half a system grows linearly with the number of particles.

In [88], these ideas are extended, and the *particle entanglement*, defined as the entanglement between two subsets of particles making up the system, is studied. The general structure of particle entanglement in many-fermion ground states, analogous to the *area law* for the more usually studied entanglement between spatial regions, is also formulated, and the basic properties of particle entanglement are uncovered by considering relatively simple itinerant models. All these ideas are widely reviewed in [89].

5. Renormalization of entanglement

A natural question arising within the study of entanglement in the quantum system is how entanglement evolves along Renormalization Group (RG) trajectories. We shall now address this issue discussing first the RG of quantum states and, then, the study of particular systems.

5.1. Renormalization of quantum states

It is customary to present RG transformations on Hamiltonians or observables. In general, a Hamiltonian is described by a set of coupling constants times operators $H = \sum_i g^i \mathcal{O}_i$. This set of operators may be infinite, including relevant, marginal and irrelevant operators, or as in the case of renormalizable quantum field theories in the continuum, it may reduce to a finite set of relevant and marginal operators. Then, upon coarse graining of short distances and an adequate rescaling, the system is described by a new set of coupling constants; the operator algebra acts as a basis. The concept of RG trajectory corresponds to analysing observables along the flow $\frac{d}{dt} = -\beta_i \frac{d}{dg^i}$, where the beta functions correspond to $\beta_i \equiv \frac{dg^i}{dt}$ are related to the change of the coupling constants as the coarse graining proceeds.

Yet, RG transformations can be understood as an action on any quantum state, regardless of its relation to any Hamiltonian [90]. Coarse graining is independent of any dynamics. This RG procedure on quantum states is not presented as common lore since explicit knowledge of e.g. the ground state of a system is not available in general. Let us address this issue.

The basic idea to perform RG on states is to produce a coarse graining of short-distance degrees of freedom, followed by a clever choice of local basis to retain the long-distance information which is retained in an optimal way. Let us take a quantum state ψ_0 and determine its RG transformed, ψ'_0 , as follows. We pairwise group the sites in the system and define a coarse-graining transformation for every pair of local d -dimensional basis states, e.g. for the sites $2j$ and $2j+1$, as $|p\rangle_{2j}|q\rangle_{2j+1} = |pq\rangle_j$. This transformation yields $\psi_0 \rightarrow \psi$. Then we have $\psi'_0 = U \otimes \dots \otimes U|\psi\rangle$, where the $d^2 \times d^2$ unitary matrix U performs the change of representative in the coarse-grained space. Note that the matrix U is non-local as seen from the $2j$ and $2j+1$ sites. Some local information is now washed out, while preserving all the quantum correlations relating the coarse-grained block to other ones.

Operators also get coarse grained along the above transformation. Take for instance an operator acting on one local Hilbert space, e.g. O_{2j} . Expectation values must remain unchanged,

$$\langle \psi_0 | O_{2j} | \psi_0 \rangle = \langle \psi'_0 | O'_j | \psi'_0 \rangle, \quad (60)$$

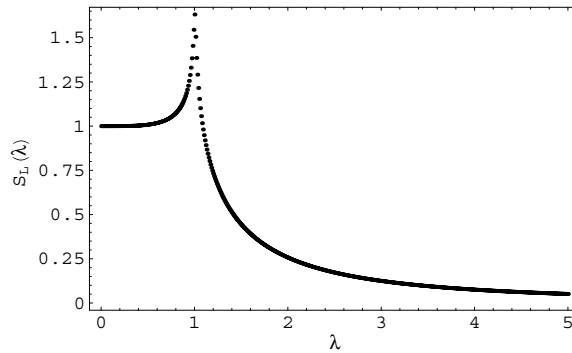


Figure 3. Entropy of entanglement is shown to decrease monotonically along the RG trajectory that takes the external magnetic field λ away from its critical value $\lambda^* = 1$. Towards the left the flow takes the system to a GHZ-line state whereas, towards the right, the system is a product state.

which leads to

$$O'_l = U(O_{2j} \otimes I_{2j+1})U^\dagger, \tag{61}$$

where I is the identity matrix. To complete an RG transformation we simply need to rescale distances, i.e. to double the lattice spacing.

This analysis can be made completely explicit in the case of states which are described as a matrix product state [90]. There, the above transformation amounts to a flow on the matrices that represent the state. In turn, a flow related to the transfer matrix can be computed. Explicit irreversibility of RG flows and the characterization of critical points followed from the flow on this transfer matrix.

5.2. Irreversibility of RG flows

Nevertheless, we return to the standard construction of RG transformations on Hamiltonians and perform a detailed study in some particular case. For instance, we may consider the quantum Ising model in a transverse field λ . It is known that the parameter λ provides a relevant deformation of the model, departing from its critical value $\lambda^* = 1$. For instance, the departure that makes $\lambda > 1$ get larger and larger corresponds to the increase of the mass of the underlying fermionic description.

An analysis of this RG trajectory can be illustrated using figure 3 (see [32, 91]). This result shows that RG trajectories are monotonically irreversible as dictated by the c-theorem in 1+1 dimensions. Furthermore, it can be seen that the ground state obeys majorization relations. That is, irreversibility is orchestrated at a very refined level, since the reshuffling of the ground state obeys an exponential set of ordering relations [44].

Irreversibility for the entanglement entropy should then be obtained as a fundamental theorem, equivalent to the c-theorem which is usually formulated in terms of the stress tensor. This was indeed done in [92]. Once, the relation between entropy and the properties of the stress tensor is made apparent.

6. Dynamics of entanglement

So far, we have studied the properties of entanglement entropy of the ground state of the system. Next, we would like to analyse how entanglement evolves in time when the system is prepared in a state that is *not* an eigenstate of the Hamiltonian.

6.1. Time evolution of the block entanglement entropy

In [93], the time evolution of the entropy of entanglement of a block of L spins in a one-dimensional system is studied. It is considered a system prepared in a pure state $|\psi_0\rangle$, which corresponds to an eigenstate of $H(\lambda_0)$ with $\lambda_0 \neq \lambda$. Then, for example, at time $t = 0$, the parameter is suddenly quenched from λ_0 to λ . In general, $|\psi_0\rangle$ will not be an eigenstate of $H(\lambda)$, and thus the system will evolve according to the equations of motion given by $H(\lambda)$. In this work, two computations are performed: one based on conformal field theory and the other on a particular solvable spin model, the Ising model. In the first case, the path integral formulation and the CFT are used in order to calculate the time evolution of the entanglement entropy of a high-energy state of the system which is not an eigenstate. Then one has to assume that the Hamiltonian is critical in order to make the theory conformally invariant. Instead, in the Ising model case, it is possible to perform calculations starting from a variety of initial states, considering both critical and non-critical regimes.

In both calculations, the entanglement entropy increases linearly with time t (after transients die away in the lattice case), up to $t^* = L/2$, in units where the maximum propagation speed of excitations is taken to be unity. For $t \gg t^*$, $S_L(t) \sim L$ saturates at an asymptotic value. This behaviour can be summarized in the following equation:

$$S_L(t) \sim \begin{cases} t & t \leq t^* \\ L & t \geq t^* \end{cases} \quad (62)$$

This behaviour of the entanglement entropy has been checked in several lattice models both analytically and numerically [72, 94–97]. In particular, in [94], the previous results are provided analytically using Toeplitz matrix representation and multidimensional phase methods for the XY model and considering large blocks.

In [93], a simple interpretation of this behaviour is proposed in terms of quasi-particle excitations emitted from the initial state at $t = 0$ and freely propagating with velocity $v \leq 1$. The idea is that at $t = 0$ and at many points of the chain, a pair of entangled quasi-particles begin to propagate in opposite directions at some constant velocity v that we will consider 1 for simplicity (see figure 4). The entanglement between the block of L spins and the rest of the system at an arbitrary time is given by the number of pairs that have one quasi-particle in the block while the other is outside. Thus, the entanglement entropy increases linearly with time until it saturates when the excitations started in the middle of the block arrive at its boundary.

All the previous results are explained in detail in the [33], where, apart from quantum quenches, a general conformal field theory approach to entanglement entropy is reviewed.

Let us point out that this increase of the entanglement entropy is unrelated to the second law of thermodynamics. Entanglement entropy can decrease or even oscillate in standard time evolution.

Let us also mention that in [72] the dynamics of entanglement was analysed for disordered systems, i.e. when the couplings between the spins take random values. In particular, the XXZ model with the couplings between the spins following a uniform distribution in the interval $[0, 1]$ was studied. It turns out that, in the presence of disorder, the entanglement entropy does not increase linearly but logarithmically. This logarithmic behaviour does not follow

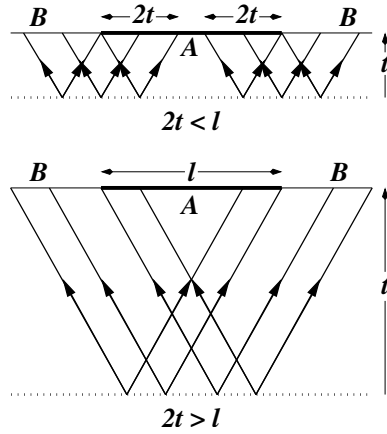


Figure 4. Schematic representation of the dynamics of block entropy. Entangled particles are emitted from the region A , and they will contribute to the block entropy as long as one of the two particles ends in the region B . Reprinted with permission from [93]. Copyright (2005) by IOP Publishing.

from an extension of the argument for the clean case assuming a diffusive propagation of the excitations, but it requires some kind of entanglement localization. This behaviour is also observed in [98] where the propagation of information through the disordered XY model is studied. In particular, both classical and quantum correlations are exponentially suppressed outside of an effective light cone whose radius grows at most logarithmically with time.

6.2. Bounds for time evolution of the block entropy

All these results are compatible with the rigorous bounds found in [99, 100] by means of the Lieb–Robinson bound [101] and its generalizations presented in [102].

The Lieb–Robinson bound states that the operator norm of the commutator of two operators O_A and O_B that act on different regions A, B of a spin network with local interactions, $h_{ij}(t)$, and in different times, verifies

$$\|[O_A(t), O_B(0)]\| \leq c N_{\min} \|O_A\| \|O_B\| e^{-\frac{L-v|t|}{\xi}}, \tag{63}$$

where L is the distance between A and B (the number of edges in the shortest path connecting A and B), $N_{\min} = \min\{|A|, |B|\}$ is the number of spins in the smallest of A or B , while $c, v, \xi > 0$ are constants depending only on $g = \max_{(i,j) \in E} \max_t \|h_{ij}(t)\|$ and the architecture of the spin lattice.

Thus, the Lieb–Robinson bound, equation (63), tells us that the norm of the commutator of two operators at different times is exponentially small outside the light cone given by the velocity v that we can understand like the speed of sound. Note that, by dimensional analysis, this velocity must be proportional to the energy scale g . It is interesting to point out that this result is also valid for the case of fermions or local Hamiltonians with exponentially decaying interactions.

In [99], it is shown, using the Lieb–Robinson bound and its generalizations [102], that correlations and information are propagated at a finite velocity in a spin network with nearest-neighbour interactions. This is a non-trivial result since in non-relativistic quantum mechanics there does not exist the notion of a light cone, i.e. local operations could be used, in principle, to send information at long distances in arbitrary short times.

Moreover, it has quantified the entanglement entropy that can be generated per unit of time between a block of spins and the rest of the system. In particular, it is found that

$$S_L(t) - S_L(0) \leq c^* g P t \tag{64}$$

where $c^* \simeq 1.9$ is a constant and P is the perimeter of the block. Finally, let us mention that all these results are complemented in [100].

6.3. Long-range interactions

The Lieb–Robinson bound is only valid for short-range interactions. Then it is interesting to study how does entanglement evolve in systems with long-range interactions. This question is addressed in [103].

In general, systems with long-range interactions are numerically intractable since, in them, the entanglement entropy scales with the volume $S_L \sim L$. Nevertheless, in [103], the interactions are restricted to Ising-type which allows us to study both the static and the dynamical entanglement properties of the system.

It is considered a lattice composed by N spins that interact according to the Hamiltonian:

$$H = \sum_{k < l} f(k, l) \frac{1}{4} (\mathbb{1} - \sigma_z^{(k)}) \otimes (\mathbb{1} - \sigma_z^{(l)}), \tag{65}$$

where the coefficients $f(k, l)$, that describe the strength of the interaction between the spins l and k , obey a distance law, that is to say, $f(k, l) = f(\|k - l\|)$.

It is assumed that the initial state is a product state of all spins pointing to the x -direction $|\Psi_0\rangle = |\rightarrow\rangle^{\otimes N}$. In order to perform the time evolution of this state, a description in terms of valence bond solids (VBS) is used (see [104]). With this method, it is possible to calculate the reduced density operator of few particles for large systems (the computational time grows linearly with the whole size of the system but exponentially with the size of the block).

In concrete, it is studied for some fixed time t the scaling properties of entanglement of a system with algebraically decaying interactions $f(k, l) = \|k - l\|^{-\alpha}$. It turns out that for $\alpha \leq 1/2$ (strong long-range interactions) the entanglement grows unbounded and the correlations do not practically decay, while for $\alpha > 1$ the system contains a bounded amount of entanglement and the correlations decay algebraically.

The dynamics of entanglement are also studied. In the limit of an infinite chain, the entanglement entropy of any block saturates for large times ($t \rightarrow \infty$) to its maximal value $S_L = L$ in a similar way as in equation (62).

7. Entanglement along quantum computation

It is known that slightly entangled quantum systems can be simulated efficiently in a classical computer [105, 106]. This implies that any quantum algorithm that would exponentially accelerate a classical computation must create, at some point, a highly entangled state. Otherwise, the quantum algorithm could be simulated efficiently in a classical computer.

Next, we want to briefly study how the entanglement evolves along a computation. In order to do this, we will consider the three most common paradigms of quantum computation: quantum circuits, adiabatic quantum computation and one-way quantum computing.

7.1. Quantum circuits

A quantum circuit is a sequence of unitary transformations (quantum gates) on a register of qubits (see [107] for a pedagogical introduction). An efficient quantum circuit is characterized

by the fact that the number of elementary gates that form it only scales polynomially in the number of qubits of the register.

The study of entanglement along a quantum circuit was addressed in [108] and [109] by means of majorization theory. In these works the introduction of entanglement in Shor’s algorithm and Grover’s algorithm was analysed respectively.

Let us recall the concept of Majorization relations, which is a more refined measure of ordering of probability distributions than the usual entropy one. We say that a probability distribution $\{p_i\}$ majorizes another probability distribution $\{q_i\}$ (written as $\vec{p} \prec \vec{q}$) if, and only if,

$$\sum_{i=1}^k p_i \leq \sum_{i=1}^k q_i, \quad k = 1, \dots, d - 1, \tag{66}$$

where $d = 2^N$ is the number of possible outcomes and it will correspond to the dimension of the Hilbert space.

These Majorization relations can be related to quantum circuits in the following way: let $|\psi_m\rangle$ be the pure state representing the register in a quantum computer in the computational basis at an operating stage labelled by $m = 0, 1, \dots, M - 1$, where M is the total number of steps of the algorithm. We can naturally associate a set of sorted probabilities $p_x^{(m)}$ corresponding to the square modulus of the coefficients of the state in the computational basis ($x \in \{|0\dots 0\rangle, |0\dots 01\rangle, \dots, |1\dots 1\rangle\}$). A quantum algorithm will be said to majorize step by step this probability distribution if

$$\vec{p}^{(m)} \prec \vec{p}^{(m+1)}, \quad \forall m = 1, \dots, M. \tag{67}$$

In such a case, there will be a neat flow of probability directed to the values of highest weight, in a way that the probability distribution will be steeper and steeper as the algorithm goes ahead. This implies that the state is becoming less entangled along the computation. Note that the majorization relations are stricter than an inequality in the entanglement entropy, in such a way that the reverse statement is not true.

In [110], the step-by-step majorization was found in the known instances of fast and efficient algorithms, namely in the quantum Fourier transform, in Grover’s algorithm, in the hidden affine function problem, in searching by quantum adiabatic evolution and in deterministic quantum walks in continuous time solving a classically hard problem. On the other hand, the optimal quantum algorithm for parity determination, which does not provide any computational speed-up, does not show step-by-step majorization.

Recently, a new class of quantum algorithms has been presented. Those are exact circuit that faithfully reproduce the dynamics of a strongly correlated many-body system. In [111], the underlying quantum circuit that reproduces the physics of the XY Hamiltonian for N spins was obtained. The philosophy inspiring that circuit was to follow the steps of the analytical solution of that integrable model. Looking at the architecture of the circuit in figure 5, it is easy to realize that the entanglement between the two sets of contiguous $N/2$ spins is transmitted through the $N/2$ SWAP gates. Therefore, the maximum entanglement entropy between these two halves of the system that this proposal may allow is $N/2$. This is because the maximum entanglement that can generate a quantum gate that acts on two qubits is 1, that is, from a product state to a maximally entangled state (Bell basis). Thus, the scaling law of the entanglement entropy that this proposal will allow will be

$$S(N/2) \leq N/2. \tag{68}$$

Note that, as we have seen in the previous sections, the entanglement entropy of the ground state in the XY model scales only logarithmically. The above circuit, then, can create much

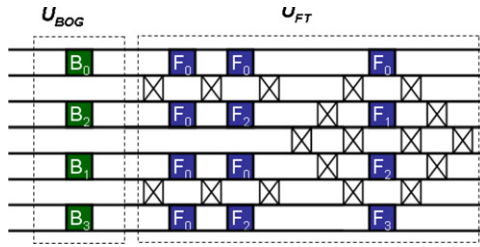


Figure 5. Structure of the quantum circuit performing the exact diagonalization of the XY Hamiltonian for eight sites. The circuit follows the structure of a Bogoliubov transformation followed by a fast Fourier transform. Three types of gates are involved: type-B (responsible for the Bogoliubov transformation and depending on the external magnetic field λ and the anisotropy parameter γ), type-fSWAP (depicted as crosses and necessary to implement the anti-commuting properties of fermions) and type-F (gates associated with the fast Fourier transform). Some initial gates have been eliminated since they only amount to some reordering of initial qubits. Reprinted with permission from Vestræte F, Cirac J I and Latorre J I 2008 *Phys. Rev. A* 79 032316. Copyright (2008) by the American Physical Society.

(This figure is in colour only in the electronic version)

more entropy than what is present in the ground state. Yet, we have also discussed the fact that time evolution does create the maximum entanglement. This, indeed, is what the above circuit achieves. This shows that the previous proposal is optimal since it carries the minimum possible number of gates such that the maximum entanglement can be created.

Let us also add a final example on exact quantum circuits. In [112], a quantum circuit that creates the Laughlin state (equation (58)) for an arbitrary number of particles (qudits) n in the case of filling fraction one is presented. The way in which entanglement grows along the circuit is also related to the amount of entanglement that each gate of the circuit can generate. In the case of this Laughlin wavefunction, the depth of the circuit grows linearly with the number of qudits, so linear entanglement $S \sim n$ can be supported by the circuit. This is precisely the entanglement that the Laughlin wavefunction with filling fraction one requires as shown in [87].

The exact circuits we have discussed (XY and Laughlin states) are both able to create linear entanglement entropy. It is then impossible that they can be simulated classically in an efficient way.

7.2. Adiabatic quantum computation

The framework of adiabatic quantum computation (AQC) was introduced in [113]. The idea of AQC is the following.

- (1) A quantum register is initially prepared on the ground state of a known initial Hamiltonian H_0 .
- (2) The system is then made to evolve adiabatically from this Hamiltonian to a new one H_P whose ground state codifies the solution to e.g. an NP-complete problem

$$H(s(t)) = (1 - s(t))H_0 + s(t)H_P. \tag{69}$$

- (3) Slow evolution from $s(t = 0) = 0$ to $s(t = T) = 1$ guarantees that the system will not jump from the instantaneous ground state of the system to the first excited state.

Quantum adiabatic computation is proven efficient provided that the minimum gap along the adiabatic evolution is only polynomially small in the number of qubits. If this was not the

case, the adiabatic computation would require an exponentially large time as measure in terms of the number of qubits in the register.

Thus, according to the previous arguments, at some point of the adiabatic evolution of a hard quantum computation the system must be highly entangled, in a similar way as it happened in the previous sections at the quantum phase transitions. This makes us expect some sort of quantum phase transition for a concrete value s_c of the Hamiltonian, point that would be characterized by a minimum energy gap.

In [114], adiabatic quantum computation is used to solve the NP-complete exact cover problem that is a particular case of the 3-SAT problem. It is defined as follows: given the n Boolean variables $\{x_i\}_{i=1,\dots,n}$, $x_i = 0, 1 \forall i$, where i is the bit index, we define a *clause* C involving the three bits i, j and k by the constraint $x_i + x_j + x_k = 1$. There are only three assignments of the set of variables $\{x_i, x_j, x_k\}$ that satisfy this equation, namely $\{1, 0, 0\}$, $\{0, 1, 0\}$ and $\{0, 0, 1\}$. An *instance* of the exact cover problem is a collection of clauses which involves different groups of three qubits. The problem is to find a string of bits $\{x_1, x_2, \dots, x_n\}$ which satisfies all the clauses.

This problem can be mapped to finding the ground state of a Hamiltonian H_P in the following way [114]: given a clause C , define the Hamiltonian associated with this clause as $H_C = \frac{1}{8}((1 + \sigma_i^z)(1 + \sigma_j^z)(1 + \sigma_k^z)(1 - \sigma_i^z)(1 - \sigma_j^z)(1 - \sigma_k^z) + (1 - \sigma_i^z)(1 - \sigma_j^z)(1 + \sigma_k^z) + (1 - \sigma_i^z)(1 + \sigma_j^z)(1 - \sigma_k^z) + (1 + \sigma_i^z)(1 - \sigma_j^z)(1 - \sigma_k^z))$, (70) where $\sigma^z|0\rangle = |0\rangle$, $\sigma^z|1\rangle = -|1\rangle$. The quantum states of the computational basis that are eigenstates of H_C with zero eigenvalue (ground states) are the ones that correspond to the bit string which satisfies C , whereas the rest of the computational states are penalized with an energy equal to 1. The problem Hamiltonian is constructed as the sum of all the Hamiltonians corresponding to all the clauses in the instance:

$$H_P = \sum_{C \in \text{instance}} H_C. \quad (71)$$

The ground state of this Hamiltonian corresponds to the quantum state whose bit string satisfies *all* the clauses.

It is known that exact cover is an NP-complete problem, so it cannot be solved in a polynomial number of steps in a classical computer [115]. This makes the exact cover problem, particularly interesting, since if we had an algorithm to efficiently solve exact cover, we could also solve all problems in the much larger NP family¹.

In [116], the evolution of the entanglement properties of the system is studied in order to see the expected sign of a quantum phase transition. There are 300 random instances for the exact cover are generated with only one possible satisfying assignment for a small number of qubits. These instances are produced by adding clauses at random until there is exactly only one satisfying assignment. In order to apply adiabatic quantum computation the initial Hamiltonian H_0 taken is a magnetic field in the x -direction

$$H_0 = \sum_{i=1}^n \frac{d_i}{2} (1 - \sigma_i^x), \quad (72)$$

where d_i is the number of clauses in which qubit i appears. Then, for each instance, the ground state is computed for several values of s of the Hamiltonian, $H(s) = (1 - s)H_0 + sH_P$, and its corresponding entanglement entropy of half a chain. The mean of the entanglement entropy over these 300 instances is performed and plotted with respect to the s parameter for different sizes of the system shown in figure 6. We can observe a peak of the entropy around the critical value $s_c \sim 0.7$.

¹ The NP problems are those whose solutions can be verified in a polynomial time.

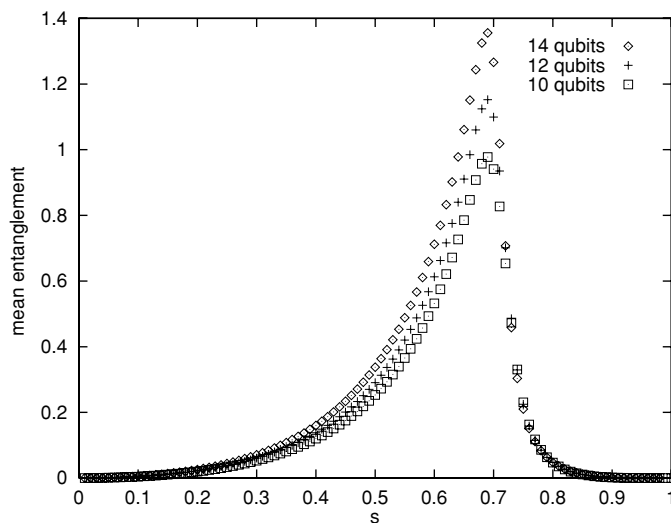


Figure 6. Average over 300 instances of the entanglement entropy between two blocks of size $n/2$ as a function of the parameter s controlling the adiabatic evolution. A peak appears for $s_c \sim 0.7$. The plot also shows the increase of the peak as the number of qubits grows $n = 10, 12, 14$. Reprinted with permission from Latorre J I and Orús R (2004) *Phys. Rev. A* **69** 062302. Copyright (2004) by the American Physical Society.

We interpret this behaviour of the entanglement entropy as follows: initially the system is in a product state and its entanglement is zero. Then the evolution makes the system explore different solutions by means of superposition states of them, that is, it becomes more and more entangled. Finally, the system throws away the bad solutions, the entanglement decreases, until the best solution is found and it rests in a product state again. Roughly speaking, the power of the quantum computer with respect to the classical one underlies in the parallelism during the computation that the superposition principle allows.

Let us make some warning remarks. The numerical simulations performed for the exact cover problem cannot determine the complexity class for the quantum algorithm. It is generally believed that quantum computers will not be able to handle NP-complete problems. Yet, the simulation shows that the best this quantum algorithm can achieved still requires a huge amount of entanglement in the register.

The divergence of the entanglement entropy that occurs at the critical point s_c has also been observed in Shor's factoring algorithm in [117], where entropy grows exponentially fast with respect to the number of qudits. This, again, makes this algorithm hard to simulate classically.

Note that in the solution of other problems, the explosion of the entropy could occur at $s_c = 1$ in such a way that the entanglement entropy was monotonically increasing. This similar behaviour of the entropy to the quantum phase transition is, therefore, problem dependent.

Recently, there has been appeared a new quantum algorithm for SAT problems that improve the previous results. It consists of a hybrid procedure that alternates non-adiabatic evolution with adiabatic steps [118].

7.3. One-way quantum computation

The one-way quantum computation (or measurement based QC) is a method used to perform quantum computation that consists of the following steps: (i) an entangled resource state is

prepared, and (ii) then single qubit measurements are performed on it. It is called ‘one-way’ because the entanglement of the state, which is the resource of the quantum computation, is destroyed by the measurements as the computation is being performed. Although the output of each individual measurement is random, they are related in such a way that the computation always succeeds. The idea is that depending on the previous outcome, one chooses the basis for the next measurements. This implies that the measurements cannot be performed at the same time.

This kind of computation was introduced in [119] where it was shown that with an initial particular state, called the cluster state, any quantum computation could be simulated. Later on, other useful states to perform one-way quantum computation were found [120–124].

The fact that the measurement-based quantum computation is universal is non-obvious, since a quantum computation is a unitary process, while a measurement is a random process. The key point is that there are two kinds of qubits in the spin system: the cluster qubits which will be measured in the process of computation and the logical qubits which constitute the quantum information that is going to be processed.

Although, globally, entanglement is expected to decrease along the quantum computation due to the single qubit measurements, in the set of logical qubits (the register that will be read out at the end of the computation), the entanglement may increase. Note that if the initial state fulfils an area law, the entanglement is enough that the register of the logical qubits is as entangled as possible. That is, the area law on a 2D state is just what is needed to have linear maximal entanglement on a register defined on a line in that state. Cluster states are just enough to handle the expected entanglement in the register. In this respect, it has been recently shown that most quantum states are too entangled to be useful in order to perform measurement-based quantum computation [125].

8. Conclusion: entanglement as the barrier for classical simulations

Entanglement is the genuine quantum property that escapes classical physics. The Hilbert space structure of a multi-partite quantum system allows for superpositions of exponentially many elements of the basis. Entropy of entanglement is a way to quantify the amount of quantum correlation between parts of such a multi-partite system. Entanglement entropy is, then, a genuine measure of the global quantumness of the state.

It serves as a conclusion to recall the deep implications of entanglement entropy in the possibility of producing faithful classical simulations of quantum mechanics. In [106], it was proven that efficient simulations are possible for any system where all its Schmidt decompositions in two arbitrary parts would carry little entropy. Therefore, entanglement is at the heart of the separation between efficient and non-efficient simulations of quantum mechanics.

What is not fully understood is what is the best strategy to classically account for quantum correlations. Two general and clever ideas are available in the literature. The first idea consists in exploiting the fact that typical interactions are local. This suggests that entanglement should be created sequentially in space from each local degree of freedom to its nearest neighbours. Then a one-dimensional state can be represented as a matrix product state which captures such a principle [126]. In higher dimensions, states can be represented as projected entangled pairs (see [127]). The second idea to classically represent quantum states as efficiently as possible consists in reconstructing the correlations in the system as a renormalization group tree. This goes under the name of multiscale entanglement renormalization ansatz (MERA), this is a more sophisticated representation which is specially suited for critical systems. The accuracy

of the approximations can be quantify using the amount of entropy of entanglement that the approximation can accommodate [128].

Multi-partite entanglement branches into many others subjects that escape this short review. Very likely, much more work is still needed to get a profound understanding of the role of entanglement in highly structured quantum systems.

Acknowledgments

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